NAMIBIA UNIVERSITY
OF SCIENCE AND TECHNOLOGY

FACULTY OF HEALTH AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

QUALIFICATION: Bachelor of Science Honours in Applied Mathematics	
QUALIFICATION CODE: 08BSMH	**LEVEL:** 8
COURSE CODE: ACA801S	**COURSE NAME:** ADVANCED COMPLEX ANALYSIS
SESSION: JUNE 2019	**PAPER:** THEORY
DURATION: 3 HOURS	**MARKS:** 100

FIRST OPPORTUNITY EXAMINATION QUESTION PAPER	
EXAMINER	PROF. G. HEIMBECK
MODERATOR:	PROF. F. MASSAMBA

INSTRUCTIONS
1. Answer ALL the questions in the booklet provided. 2. Show clearly all the steps used in the calculations. 3. All written work must be done in blue or black ink and sketches must be done in pencil.

PERMISSIBLE MATERIALS

1. Non-programmable calculator without a cover.

THIS QUESTION PAPER CONSISTS OF 4 PAGES (Including this front page)

Question 1 [16 marks]

a) Consider $\mathbb{C}$ as a vector space over $\mathbb{R}$. Let $a \in \mathbb{C}$. What is $[a]$ by definition? Prove that

$$[a] = \{\lambda a | \ \lambda \in \mathbb{R}\}.$$

[6]

b) What is a line of the complex plane? State the definition. [3]

c) Let $a, b \in \mathbb{C}$ such that $a \neq b$. Prove that there exists exactly one line L of the complex plane such that $a, b \in L$. [7]

Question 2 [17 marks]

Consider $\mathbb{C}$ with its standard topology.

a) Let $a \in \mathbb{C}$ and $\varepsilon > 0$. You are reminded that

$$N_\varepsilon(a) := \{z \in \mathbb{C} | \ |z - a| < \varepsilon\}.$$

Now prove that $N_\varepsilon(a) \cap \mathbb{R}$ is an open interval of $\mathbb{R}$. [6]

b) What is the topology of the subspace $\mathbb{R}$ of $\mathbb{C}$? State the definition. [3]

c) Show that the subspace $\mathbb{R}$ of $\mathbb{C}$ is the real line. [8]

Question 3 [14 marks]

Let $\sum a_k(z - c)^k$ be a convergent power series and $\varepsilon > 0$ such that $N_\varepsilon(c)$ is contained in the set of convergence of the power series. Let $f : N_\varepsilon(c) \to \mathbb{C}$ be defined by

$$f(z) := \sum_{k=0}^{\infty} a_k(z - c)^k.$$

a) Prove that f is n-times differentiable for all $n \in \mathbb{N}$ and

$$f^{(n)}(z) = \sum_{k=0}^{\infty} (k + n)(k + n - 1) \cdot \ldots \cdot (k + 1) a_{k+n}(z - c)^k,$$

for all $n \in \mathbb{N}$ and all $z \in N_\varepsilon(c)$. With respect to differentiability, what kind of function is f? [7]

b) Show that

$$\frac{f^{(n)}(c)}{n!} = a_n, \quad \text{for all } n \in \mathbb{N}_0.$$

What does this mean for the power series? [5]

c) What is the Taylor series of f at c? [2]

Question 4 [12 marks]

Let $X \subset \mathbb{C}$ and let $(f_n)_{\mathbb{N}}$ be a sequence of complex-valued functions on X,

a) State the definition of the limit function of $(f_n)_{\mathbb{N}}$. [2]

b) When does $(f_n)_{\mathbb{N}}$ uniformly converge on X? State the definition. [3]

c) Now assume that $(f_n)_{\mathbb{N}}$ converges uniformly on X to the limit function f. Let $(z_n)_{\mathbb{N}}$ be a sequence in X which converges to $w \in X$. Show that $(f_n(z_n) - f(z_n))_{\mathbb{N}}$ is a null sequence. If f is continuous at w prove that $(f_n(z_n))_{\mathbb{N}}$ converges to $f(w)$. [7]

Question 5 [13 marks]

a) State Cauchy's integral formula for a disc. [3]

b) i) Let $O \subset \mathbb{C}$ be open and let $f: O \to \mathbb{C}$ be a holomorphic function. Let $a \in O$ and $\varepsilon > 0$ such that $\overline{N_\varepsilon(a)} \subset O$. Show that

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_\varepsilon(a)} \frac{f(\zeta)}{(\zeta - a)^{k+1}} \, d\zeta \right) (z-a)^k$$

for all $z \in N_\varepsilon(a)$. [7]

ii) Conclude that f is infinitely differentiable. [3]

Question 6 [14 marks]

Let $O \subset \mathbb{C}$ be open and let $f: O \to \mathbb{C}$ be a holomorphic function.

a) What is an isolated singularity of f? State the definition. [3]

b) When is $c \in \mathbb{C}$ a removable singularity of f? How does one remove such a singularity? [6]

c) What is a pole of f? What is the order of a pole? [5]

Question 7 [14 marks]

Let $f: \mathbb{C} - \{0\} \to \mathbb{C}$ be defined by

$$f(z) := e^{-\frac{1}{z}}.$$

a) Make the Laurent expansion of f at 0 and find the regular and principal part of the Laurent series. [3]

b) What kind of singularity is 0? How does f behave in the vicinity of 0? [5]

c) Find

$$\int_{C_1(0)} e^{-\frac{1}{\zeta}}\, d\zeta.$$

[6]

End of the question paper